Non-standard connections in classical mechanics

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Abstract

In the jet-bundle description of first-order classical field theories there are some elements, such as the lagrangian energy and the construction of the hamiltonian formalism, which require the prior choice of a connection. Bearing these facts in mind, we analyze the situation in the jet-bundle description of time-dependent classical mechanics. So we prove that this connection-dependence also occurs in this case, although it is usually hidden by the use of the "natural" connection given by the trivial bundle structure of the phase spaces in consideration. However, we also prove that this dependence is dynamically irrelevant, except where the dynamical variation of the energy is concerned. In addition, the relationship between first integrals and connections is shown for a large enough class of lagrangians.

Key words: Connections, fiber bundles, lagrangian and hamiltonian formalisms.

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1 Introduction

One of the most interesting lines of current research in mathematical physics is the geometric formulation of first-order classical field theories, which is achieved mainly using jet bundles $J^1E \to E \to M$ and the geometrical structures with which they are endowed [2], [3], [6], [8], [9], [10], [11], [12], [14], [15], [16], [18].

Among all the relevant features observed when dealing with these geometrical formulations, we wish to point out the following: there are some dynamical elements of the theories depending on the prior choice of a connection in the configuration bundle $\pi\colon E \to M$: For instance:

- In the lagrangian formalism, the definition of the *density of lagrangian energy* and the *lagrangian energy function* [8].

- The construction of the hamiltonian formalism of these theories, in particular the *hamiltonian function* and the evolution equations (see, for instance, [3]).

It is also known that time-dependent mechanical systems can be geometrically described using jet bundles. Then $M = \mathbf{R}$, $E = Q \times \mathbf{R}$ and $J^1E = \mathrm{T}Q \times \mathbf{R}$ (where Q represents the configuration space of the system), and consequently this can be considered as a particular situation of field theories [7], [9]. In this context, since $Q \times \mathbf{R}$ is a trivial bundle, it is canonically endowed with a "natural" connection which is used (when necessary) to define all the dynamical and geometric elements of the theory; in particular, the hamiltonian formalism and the energy lagrangian function. A possible conclusion of this approach is that, for non-autonomous mechanical systems, the definition of these dynamical elements does not depend on the choice of any connection.

The aim of this paper is to make evident the influence of the choice of a connection in the geometrical construction of some elements of the theory. In order to achieve this, we will choose an arbitrary connection in the bundle $Q \times \mathbf{R} \to \mathbf{R}$ and we will re-construct the dynamics of the theory starting from this point. In addition, we will study the relation among the descriptions coming from different choices of a connection, and we will interpret the results of this analysis from a dynamical point of view. Connections on $Q \times \mathbf{R} \to \mathbf{R}$ can be understood as special time-actions in the manifold $Q \times \mathbf{R}$. The standard one is by translations $(q, t) \overset{s}{\mapsto} (q, t + s)$. Other actions correspond to non-standard connections. In this sense changes in connections imply changes in the energy and in the geometric elements of the theory.

The structure of the work is as follows:

First, we introduce the basic ideas about connections in the bundle $Q \times \mathbf{R}$ and the natural geometric elements in the bundle $\mathrm{T}Q \times \mathbf{R} \to Q \times \mathbf{R}$. Section 3 is devoted to presenting the lagrangian formalism (using arbitrary connections) and showing which of its elements are connection-depending. We obtain results on the variation of the energy along the motion of the system and on the relationship between first integrals and connections. All these results generalize classical ones for non-autonomous systems. In the fourth section we construct the hamiltonian formalism for non-autonomous systems, depending on the choice of an arbitrary connection and, subsequently, we give a dynamical interpretation of the results so obtained. In section 5 a characterization of the lagrangian energy function, based on variational principles, is given. A final section is devoted to summarizing the conclusions reached in the work.

All the manifolds and maps are C^∞. Sum over repeated indices is understood.

2 The 1-jet bundle of $\pi\colon Q \times \mathbf{R} \to \mathbf{R}$. Geometric structures and connections

The ideas in this section are known. We merely emphasize the differences between the general situation and this particular one in order to make the paper more readable and selfcontained. See [13] and [19] as general references.

2.1 Connections in $\pi\colon Q \times \mathbf{R} \to \mathbf{R}$

Consider the bundle $\pi\colon Q \times \mathbf{R} \to \mathbf{R}$, where Q is an n-dimensional differentiable manifold (the *configuration space* of a physical system). The 1-jet bundle of sections of π is $\pi_1\colon TQ \times \mathbf{R} \to Q \times \mathbf{R}$. In fact, if $\phi = (\phi_Q, \mathrm{Id})$ is a local section of π defined in a neighbourhood of $s \in \mathbf{R}$, with $\phi(s) = (q, s)$, then the 1-jet equivalence class of ϕ is determined by $(T_s\phi_Q)\left(\dfrac{\mathrm{d}}{\mathrm{d}t}\right)$; that is, an element of T_qQ. Conversely, if $v \in T_qQ$ and $s \in \mathbf{R}$, there is an equivalence class of curves $\phi_Q\colon (s - \epsilon, s + \epsilon) \to Q$ with $T_s\phi_Q\left(\dfrac{\mathrm{d}}{\mathrm{d}t}\right) = v$; so a 1-jet equivalence class of local sections is defined and $\phi = (\phi_Q, \mathrm{Id})$ is one of its representatives.

If $(U; q^\mu)$ is a local chart in Q, then a local chart in $TQ \times \mathbf{R}$ is $(\pi_1^{-1}(U \times \mathbf{R}); q^\mu, t, v^\mu)$ where

$$v^\mu((q, s), v) = \left(\frac{\mathrm{d}\phi^\mu(t)}{\mathrm{d}t}\right)_s$$

$\phi\colon \mathbf{R} \to Q \times \mathbf{R}$ being a representative of $((q, s), v) \in TQ \times \mathbf{R}$ and $\phi^\mu = q^\mu \circ \phi_Q$. Of course, these coordinates v^μ are the physical velocities.

In order to introduce the connections, we must study the tangent bundle of $Q \times \mathbf{R}$. Observe that there is a natural identification between $T(Q \times \mathbf{R})$ and $TQ \times T\mathbf{R} := \pi_Q^*TQ \oplus \pi^*T\mathbf{R}$ given by

$$\begin{array}{rccl}
\psi & : & T(Q \times \mathbf{R}) & \longrightarrow \quad TQ \times T\mathbf{R} \\
& & ((q, s), u) & \longmapsto \quad ((q, s), T_{(q,s)}\pi_Q(u) + T_{(q,s)}\pi(u))
\end{array}$$

where $\pi_Q\colon Q \times \mathbf{R} \to Q$ is the natural projection.

But π_Q^*TQ is identified with $V(\pi)$ (the vertical subbundle of $T(Q \times \mathbf{R})$ with respect to π). In fact, if $(q_o, s) \in Q \times \mathbf{R}$ and $j_s\colon Q \to Q \times \mathbf{R}$ is the s-injection defined by $j_s(q) = (q, s)$, then $V_{(q_o,s)}(\pi) = T_{q_o}j_s(T_{q_o}Q)$. So we have the natural spitting

$$T(Q \times \mathbf{R}) = V(\pi) \oplus \pi^*T\mathbf{R}$$

and $\pi^*T\mathbf{R}$ is called the *horizontal subbundle*. As a consequence, if $v \in T_{(q,s)}(Q \times \mathbf{R})$, we will write $v = v_Q + v_{\mathbf{R}}$ in this splitting.

This natural splitting will be called the *standard connection* in the bundle $\pi\colon Q \times \mathbf{R} \to \mathbf{R}$. The theory of connections describes the possible splittings of this kind.

Taking into account this model and following [19], we prove the following statement:

Proposition 1 *The following elements on $\pi\colon Q \times \mathbf{R} \to \mathbf{R}$ can be canonically constructed one from the other:*

1. *A section of $\pi_1 \colon TQ \times \mathbf{R} \to Q \times \mathbf{R}$, that is a mapping $\nabla \colon Q \times \mathbf{R} \to TQ \times \mathbf{R}$ such that $\pi_1 \circ \nabla = \mathrm{Id}_{Q \times \mathbf{R}}$.*

2. *A subbundle $\mathrm{H}(\nabla)$ of $\mathrm{T}(Q \times \mathbf{R})$ such that*

$$\mathrm{T}(Q \times \mathbf{R}) = \mathrm{V}(\pi) \oplus \mathrm{H}(\nabla) \tag{1}$$

3. *A semibasic 1-form $\tilde{\nabla}$ on $Q \times \mathbf{R}$ with values in $\mathrm{T}(Q \times \mathbf{R})$ (that is, an element of $\Gamma(Q \times \mathbf{R}, \pi^* \mathrm{T}^* Q) \otimes \aleph(Q \times \mathbf{R})$), such that $\alpha \circ \tilde{\nabla} = \alpha$, for every semibasic form $\alpha \in \Omega^1(Q \times \mathbf{R})$.*

(We use the notation $\Gamma(A, B)$ for the set of sections of the bundle $B \to A$).

(*Proof*) $(1 \Rightarrow 3)$ Let ∇ be a section of π_1 and $\phi = (\phi_Q, \mathrm{Id})$ a representative of $\nabla(q, s) \in \mathrm{T}Q \times \mathbf{R}$. We have the map

$$\mathrm{T}_s \phi \circ \mathrm{T}_{(q,s)} \pi \colon \mathrm{T}_{(q,s)}(Q \times \mathbf{R}) \longrightarrow \mathrm{T}_{(q,s)}(Q \times \mathbf{R})$$

Let $\tilde{\nabla} \colon \mathrm{T}(Q \times \mathbf{R}) \longrightarrow \mathrm{T}(Q \times \mathbf{R})$ be the map defined in this way for every $(q, s) \in Q \times \mathbf{R}$. Obviously $\tilde{\nabla}$ vanishes on the π-vertical vector fields.

In order to prove the differentiability of $\tilde{\nabla}$, take a local chart (q^μ, t, v^μ). The local expression of ∇ is $\nabla(q, s) = (q, s, \gamma^\mu(q, s))$. Then, if $X = f \dfrac{\partial}{\partial t} + g^\mu \dfrac{\partial}{\partial q^\mu} \in \aleph(Q \times \mathbf{R})$, we have

$$\tilde{\nabla}((q, s); X) = f(q, s) \left(\frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu} \right) \Big|_{(q,s)}$$

which is a differentiable vector field in $Q \times \mathbf{R}$. So $\tilde{\nabla} \in \Gamma(Q \times \mathbf{R}, \pi^* \mathrm{T} Q) \otimes \aleph(Q \times \mathbf{R})$.

The local expression of $\tilde{\nabla}$ is

$$\tilde{\nabla} = \mathrm{d}t \otimes \left(\frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu} \right) \tag{2}$$

Then, if $\alpha \in \Omega^1(Q \times \mathbf{R})$ is a semibasic form, then the relation $\alpha \circ \tilde{\nabla} = \alpha$ holds, as was desired.

$(3 \Rightarrow 1)$ Given $\tilde{\nabla}$, from its local expression 2, we can construct local maps $\nabla \colon U \times \mathbf{R} \to \mathrm{T}U \times \mathbf{R}$ given by $\nabla(q, s) := (q, s, \gamma^\mu(q, s))$, for every $(q, s) \in V = U \times \mathbf{R}$. But, since $\tilde{\nabla}$ is globally defined, its local representations agree on the intersections of local charts. In the same way, the local expressions of ∇ can be glued and we get a globally defined section of π_1. Moreover, if $\tilde{\nabla}_1 \neq \tilde{\nabla}_2$, then they have different local representations on the neighbourhood of a point, and hence they induce different sections ∇_1 and ∇_2.

$(3 \Rightarrow 2)$ Given $\tilde{\nabla}$, since it is a semibasic form we have $\tilde{\nabla} \circ \tilde{\nabla} = \tilde{\nabla}$. Then we have the splitting

$$\mathrm{T}(Q \times \mathbf{R}) = \mathrm{Ker}(\tilde{\nabla}) \oplus \mathrm{Im}(\tilde{\nabla})$$

But $\mathrm{Ker}(\tilde{\nabla}) = \mathrm{V}(\pi)$, and we have the splitting (1) taking $\mathrm{H}(\nabla) = \mathrm{Im}(\tilde{\nabla})$.

$(2 \Rightarrow 3)$ Given the subbundle $\mathrm{H}(\nabla)$ and the splitting (1), we have the maps

$$h_\nabla \colon \mathrm{T}(Q \times \mathbf{R}) \to \mathrm{H}(\nabla) \ , \ v_\nabla \colon \mathrm{T}(Q \times \mathbf{R}) \to \mathrm{V}(\pi)$$

called the *horizontal* and *vertical* projections (we will use the same symbols h_∇ and v_∇ for the natural extensions of these maps to vector fields). Then we can define a form $\tilde{\nabla}$ with values in $\mathrm{T}(Q \times \mathbf{R})$ by

$$\tilde{\nabla}((q, s); u) := h_\nabla((q, s), u)$$

Obviously, $\tilde{\nabla}$ is semibasic and $\alpha \circ \tilde{\nabla} = \alpha$ for every semibasic form.   ∎

Definition 1 *A connection* in the bundle $\pi\colon Q \times \mathbf{R} \to \mathbf{R}$ *is one of the above mentioned equivalent elements.*

Then $\mathrm{H}(\nabla)$ *is called the* horizontal subbundle *of* $\mathrm{T}(Q \times \mathbf{R})$ *associated with the connection* ∇ *and its sections* horizontal vector fields. $\tilde{\nabla}$ *is called the* connection form.

In a local chart (q^μ, t, v^μ), the local expressions of these elements are

$$\tilde{\nabla} = \mathrm{d}t \otimes \left(\frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu} \right) \quad , \quad \mathrm{H}(\nabla) = \operatorname{span} \left\{ \frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu} \right\}$$

For every vector field $X \equiv X_\mathbf{R} + X_Q \equiv f \dfrac{\partial}{\partial t} + X_Q \in \aleph(Q \times \mathbf{R})$ (where $X_Q \in \mathrm{C}^\infty(Q \times \mathbf{R}) \otimes \aleph(Q)$), the horizontal and vertical projections are given by

$$X_{h_\nabla} \equiv h_\nabla(X) \;=\; f \left(\frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu} \right)$$

$$X_{v_\nabla} \equiv v_\nabla(X) \;=\; X_Q - f \gamma^\mu \frac{\partial}{\partial q^\mu}$$

and we have the splitting

$$X = X_Q + f \frac{\partial}{\partial t} = X - \tilde{\nabla}(X) + \tilde{\nabla}(X) = X_{v_\nabla} + X_{h_\nabla}$$

Moreover, we have the following result:

Proposition 2 *Every connection in the bundle* $\pi\colon Q \times \mathbf{R} \to \mathbf{R}$ *induces a canonical lifting*

$$\begin{aligned} \aleph(\mathbf{R}) &\longrightarrow \aleph(Q \times \mathbf{R}) \\ X &\longmapsto \bar{X} \end{aligned}$$

where $\bar{X}$ *is a horizontal vector field.*

(*Proof*) Let ∇ be a connection, $(q, s) \in Q \times \mathbf{R}$ and and $\phi = (\phi_Q, \mathrm{Id})$ a representative of $\nabla(q, s)$. If $X \in \aleph(\mathbf{R})$, we define $\bar{X}$ by

$$\tilde{X}(q, s) := \mathrm{T}_s \phi(X_s)$$

From the local expression of ∇ we deduce that $\bar{X}$ is a C^∞-vector field and it is horizontal because

$$\tilde{\nabla}((q, s), \bar{X}(q, s)) = (\mathrm{T}_s \phi \circ \mathrm{T}_{(q,s)} \pi)(\bar{X}_{(q,s)}) = (\mathrm{T}_s \phi \circ \mathrm{T}_{(q,s)} \pi \circ \mathrm{T}_s \phi)(X_s) = \mathrm{T}_s \phi(X_s) = \bar{X}_{(q,s)}$$

■

But $\aleph(\mathbf{R})$ has a global generator, $\dfrac{\mathrm{d}}{\mathrm{d}t}$; so, given a connection ∇, we can take its lifting:

$$\overline{\frac{\mathrm{d}}{\mathrm{d}t}}\bigg|_{(q,s)} = \mathrm{T}_s \phi_Q \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_s \right) + \mathrm{T}_s \mathrm{Id}_q \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_s \right) = \mathrm{T}_s \phi_Q \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_s \right) + \frac{\partial}{\partial t}\bigg|_{(q,s)}$$

Observe that $\mathrm{T}_s \phi_Q \left(\dfrac{\mathrm{d}}{\mathrm{d}t}\bigg|_s \right)$ is a section of the bundle $\pi_Q^* \mathrm{T}Q \to Q \times \mathbf{R}$; that is, a time-dependent vector field in Q. Then we have:

Proposition 3 *A connection in the bundle $\pi: Q \times \mathbf{R} \to \mathbf{R}$ is equivalent to a time-dependent vector field in Q.*

(*Proof*)   Given a connection ∇, taking $Y = \dfrac{\overline{\mathrm{d}}}{\mathrm{d}t} - \dfrac{\partial}{\partial t}$, we have the desired vector field.

Conversely, given a time-dependent vector field $Y: Q \times \mathbf{R} \to \pi_Q^* \mathrm{T}Q$, we have a connection defined by

$$
\begin{array}{rccc}
\nabla & : & Q \times \mathbf{R} & \longrightarrow & \mathrm{T}Q \times \mathbf{R} \\
 & & (q, s) & \mapsto & (Y_{(q,s)}, s)
\end{array}
$$

∎

If Y is the vector field induced by ∇, then the connection form is written as

$$
\tilde{\nabla} = \mathrm{d}t \otimes \left(\frac{\partial}{\partial t} + Y \right)
$$

As we have pointed out above, the trivial bundle $\pi: Q \times \mathbf{R} \to \mathbf{R}$ has a natural connection: the standard one ∇_0, with $\mathrm{H}(\nabla_0) = \pi^* \mathrm{T}\mathbf{R}$. In this case $\tilde{\nabla}_0 = \mathrm{d}t \otimes \dfrac{\partial}{\partial t}$, the associated time-dependent vector field is $Y_0 = 0$ and the lifting induced by ∇_0 is given by $\dfrac{\overline{\mathrm{d}}}{\mathrm{d}t}\bigg|_{(q,s)} = \dfrac{\partial}{\partial t}\bigg|_{(q,s)}$.

If we have another connection ∇ with associated vector field Y, this lifting is

$$
\frac{\overline{\mathrm{d}}}{\mathrm{d}t}\bigg|_{(q,s)} = \frac{\partial}{\partial t}\bigg|_{(q,s)} + Y_{(q,s)}
$$

then we can understand that the "lines of time" induced by this connection are the integral curves of the vector field $\dfrac{\partial}{\partial t} + Y := \tilde{Y}$, which is called the *suspension of Y* [1].

From now on, we will refer to *non-standard connections* in the bundle $\pi: Q \times \mathbf{R}$ in those cases that differ from the standard one.

2.2   Geometric elements

In the lagrangian formalism the dynamics takes place in the manifold $\mathrm{T}Q \times \mathbf{R}$. Then, in order to set it, we need to introduce some geometrical elements of the bundle $\pi_1 \mathrm{T}Q \times \mathbf{R} \to Q \times \mathbf{R}$ (see [7], [9], [11] and [19] for details).

The structural canonical 1-form [9]

We can define a 1-form ϑ in $\mathrm{T}Q \times \mathbf{R}$, with values in $\pi_1^* \mathrm{V}(\pi)$, in the following way:

$$
\vartheta(((q,s),u); X) := (\mathrm{T}_{((q,s),u)}\pi_1 - \mathrm{T}_{((q,s),u)}(\phi \circ \pi \circ \pi_1))(X_{(q,s)})
$$

where ϕ is a representative of $((q,s),u) \in \mathrm{T}Q \times \mathbf{R}$.

ϑ is called the *structure canonical form* of $\mathrm{T}Q \times \mathbf{R}$. Its local expression is

$$
\vartheta = (\mathrm{d}q^\mu - v^\mu \mathrm{d}t) \otimes \frac{\partial}{\partial q^\mu}
$$

The vertical endomorphisms.

Taking into account that $\pi_1 \colon \mathrm{T}Q \times \mathbf{R} \to Q \times \mathbf{R}$ is a vector bundle and the fiber on $(q, s) \in Q \times \mathbf{R}$ is $\mathrm{T}_q Q \times \{s\}$, there exists a canonical diffeomorphism between the π_1-vertical subbundle and $\pi_1^*(\mathrm{T}Q \times \mathbf{R})$, that is,

$$\mathrm{V}(\pi_1) \simeq \pi_1^*(\mathrm{T}Q \times \mathbf{R}) \simeq \pi_1^* \pi_Q^* \mathrm{T}Q \simeq \pi_1^* \mathrm{V}(\pi)$$

We denote by

$$\mathcal{S} \colon \pi_1^* \mathrm{V}(\pi) \to \mathrm{V}(\pi_1)$$

the realization of this isomorphism and we will use the same notation $\mathcal{S}$ for its action on the modules of sections of these bundles.

If (q^μ, t, v^μ) is a local system of canonical coordinates in $\mathrm{T}Q \times \mathbf{R}$, by construction we have that $\mathcal{S}\left(\dfrac{\partial}{\partial q^\mu}\right) = \dfrac{\partial}{\partial v^\mu}$, and so

$$\mathcal{S} = \xi^\mu \otimes \frac{\partial}{\partial v^\mu}$$

where $\{\xi^\mu\}$ is the dual basis of $\left\{\dfrac{\partial}{\partial q^\mu}\right\}$ in $\pi_1^* \mathrm{V}(\pi)$.

$\mathcal{S}$ is an element of $\Gamma(\mathrm{T}Q \times \mathbf{R}, \pi_1^* \mathrm{V}^*(\pi)) \otimes \Gamma(\mathrm{T}Q \times \mathbf{R}, \mathrm{V}(\pi_1))$. Taking into account that the structure form ϑ is an element of $\Omega^1(\mathrm{T}Q \times \mathbf{R}, \pi_1^* \mathrm{V}(\pi_1)) = \Omega^1(\mathrm{T}Q \times \mathbf{R}) \otimes \Gamma(\mathrm{T}Q \times \mathbf{R}, \pi_1^* \mathrm{V}(\pi))$, using the natural duality, by contracting $\mathcal{S}$ with ϑ we obtain an element

$$\mathcal{V} := i(\mathcal{S})\vartheta \in \Omega^1(\mathrm{T}Q \times \mathbf{R}) \otimes \Gamma(\mathrm{T}Q \times \mathbf{R}, \mathrm{V}(\pi_1))$$

whose local expression is

$$\mathcal{V} = (\mathrm{d}q^\mu - v^\mu \mathrm{d}t) \otimes \frac{\partial}{\partial v^\mu}$$

Notice that $\mathcal{V}$ can be thought as a $\mathrm{C}^\infty(\mathrm{T}Q \times \mathbf{R})$-module morphism $\mathcal{V} \colon \aleph(\mathrm{T}Q \times \mathbf{R}) \to \aleph(\mathrm{T}Q \times \mathbf{R})$ with image on the π_1-vertical vector fields.

$\mathcal{S}$ and $\mathcal{V}$ are called the *vertical endomorphisms* of $\mathrm{T}Q \times \mathbf{R}$.

Observe that $\mathcal{S}$ and $\mathcal{V}$ are sections of different bundles. However, if we have a connection ∇, $\mathcal{S}$ can be also understood as an endomorphism on $\aleph(\mathrm{T}Q \times \mathbf{R})$. In fact, notice that the splitting $\mathrm{T}(Q \times \mathbf{R}) = \mathrm{V}(\pi) \oplus \mathrm{H}(\nabla)$, induced by the connection ∇, has a dual counterpart $\mathrm{T}^*(Q \times \mathbf{R}) = \mathrm{V}^*(\pi) \oplus \mathrm{H}^*(\nabla)$. So, $\mathrm{V}^*(\pi)$ is identified by ∇ with a subbundle of $\mathrm{T}^*(Q \times \mathbf{R})$. With the same notation as above, if $\{\xi^\mu\}$ is the dual basis of $\left\{\dfrac{\partial}{\partial q^\mu}\right\}$ in $\pi_1^* \mathrm{V}(\pi)$, then this identification is given by

$$\xi^\mu \mapsto \mathrm{d}q^\mu - \gamma^\mu \mathrm{d}t$$

if $\nabla = \mathrm{d}t \otimes \left(\dfrac{\partial}{\partial t} + \gamma^\mu \dfrac{\partial}{\partial q^\mu}\right)$.

In the same way, we have the splitting

$$\pi_1^* \mathrm{T}^*(Q \times \mathbf{R}) = \pi_1^* \mathrm{V}^*(\pi) \oplus \pi_1^* \mathrm{H}^*(\nabla)$$

and the injection $j_\nabla \colon \pi_1^* \mathrm{V}^*(\pi) \hookrightarrow \pi_1^* \mathrm{T}^*(Q \times \mathbf{R})$. But this last bundle is a subbundle of $\mathrm{T}^*(\mathrm{T}Q \times \mathbf{R})$ whose sections are the π_1-semibasic 1-forms in $\mathrm{T}Q \times \mathbf{R}$. Hence, by means of this injection, $\mathcal{S}$ is an element of $\Omega^1(\mathrm{T}Q \times \mathbf{R}) \otimes \aleph(\mathrm{T}Q \times \mathbf{R})$, with values in the vertical vector fields, which will be denoted by $\mathcal{S}^\nabla$. Its local expression is

$$\mathcal{S}^\nabla = (\mathrm{d}q^\mu - \gamma^\mu \mathrm{d}t) \otimes \frac{\partial}{\partial v^\mu}$$

Now, we are able to consider the difference $\mathcal{S}^{\nabla} - \mathcal{V} \in \Omega^1(\mathrm{T}Q \times \mathbf{R}) \otimes \aleph(\mathrm{T}Q \times \mathbf{R})$ whose local expression is

$$\mathcal{S}^{\nabla} - \mathcal{V} = (v^{\mu} - \gamma^{\mu})\mathrm{d}t \otimes \frac{\partial}{\partial t}$$

which will be used henceforth in order to characterize the lagrangian energy.

3 Lagrangian formalism. Connections and lagrangian energy functions

A *time-dependent lagrangian function* is a function $\mathcal{L} \in \mathrm{C}^{\infty}(\mathrm{T}Q \times \mathbf{R})$. As it is known, we can construct the lagrangian forms associated with $\mathcal{L}$ using the geometrical structure of $\mathrm{T}Q \times \mathbf{R}$.

Definition 2 *The* Poincaré-Cartan 1 and 2-forms *associated with the lagrangian function $\mathcal{L}$ are the forms in $\mathrm{T}Q \times \mathbf{R}$ defined by*

$$\Theta_{\mathcal{L}} := \mathrm{d}\mathcal{L} \circ \mathcal{V} + \mathcal{L}\mathrm{d}t \quad , \quad \Omega_{\mathcal{L}} := -\mathrm{d}\Theta_{\mathcal{L}}$$

For the coordinate expressions of the Poincaré-Cartan forms we obtain

$$\begin{aligned}
\Theta_{\mathcal{L}} &= \frac{\partial \mathcal{L}}{\partial v^{\mu}}(\mathrm{d}q^{\mu} - v^{\mu}\mathrm{d}t) + \mathcal{L}\mathrm{d}t = \left(\mathcal{L} - v^{\mu}\frac{\partial \mathcal{L}}{\partial v^{\mu}}\right)\mathrm{d}t + \frac{\partial \mathcal{L}}{\partial v^{\mu}}\mathrm{d}q^{\mu} \\
\Omega_{\mathcal{L}} &= -\mathrm{d}\left(\frac{\partial \mathcal{L}}{\partial v^{\mu}}\right) \wedge \mathrm{d}q^{\mu} + \mathrm{d}\left(\frac{\partial \mathcal{L}}{\partial v^{\mu}}v^{\mu} - \mathcal{L}\right) \wedge \mathrm{d}t
\end{aligned}$$

Observe that these elements do not depend on the connection.

As usual, we say that a lagrangian function $\mathcal{L}$ is *regular* iff its associated form $\Omega_{\mathcal{L}}$ has maximal rank, which is equivalent to demanding that $\det\left(\dfrac{\partial^2 \mathcal{L}}{\partial v^{\mu}\partial v^{\nu}}\right)$ is different from zero at every point.

Assuming the regularity of $\mathcal{L}$, the dynamics of the system is described by a vector field $X_{\mathcal{L}} \in \mathcal{X}(\mathrm{T}Q \times \mathbf{R})$, which is a *Second Order Differential Equation* (SODE), such that:

$$i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0 \quad , \quad i(X_{\mathcal{L}})\mathrm{d}t = 1 \tag{3}$$

As a consequence, the integral curves of $X_{\mathcal{L}}$ verify the *Euler-Lagrange equations* and $X_{\mathcal{L}}$ is obviously independent of the connection.

A very different picture arises when we try to define intrinsically the *lagrangian energy function*. If we consider the standard connection, it can be obtained as follows: take the lifting of $\dfrac{\mathrm{d}}{\mathrm{d}t}$ from $\mathbf{R}$ to $\mathrm{T}Q \times \mathbf{R}$ given by the connection, which will be denoted as usual by $\dfrac{\partial}{\partial t}$, then

$$\mathrm{E}_{\mathcal{L}} = -i\left(\frac{\partial}{\partial t}\right)\Theta_{\mathcal{L}}$$

being $\mathrm{E}_{\mathcal{L}} = \dfrac{\partial \mathcal{L}}{\partial v^{\mu}}v^{\mu} - \mathcal{L}$ its local expression.

Of course, this changes if we consider a non-standard connection ∇. Then it is more appropriate to define the lagrangian energy function associated to ∇ in the following way:

Definition 3 *Let ∇ be a connection in $\pi\colon Q \times \mathbf{R} \to \mathbf{R}$, $\tilde{Y} \in \aleph(Q \times \mathbf{R})$ the lifting of $\dfrac{\mathrm{d}}{\mathrm{d}t}$ induced by ∇ and $j^1\tilde{Y} \in \aleph(\mathrm{T}Q \times \mathbf{R})$ its canonical lifting. The* lagrangian energy function *associated with the lagrangian function $\mathcal{L}$ and the connection ∇ is*

$$\mathrm{E}_{\mathcal{L}}^{\nabla} = - i(j^1\tilde{Y})\Theta_{\mathcal{L}}$$

In a local system of coordinates we have

$$j^1\tilde{Y} = \frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu} + \left(\frac{\partial \gamma^\mu}{\partial t} + v^\nu \frac{\partial \gamma^\mu}{\partial q^\nu} \right) \frac{\partial}{\partial v^\mu}$$

and

$$\mathrm{E}_{\mathcal{L}}^{\nabla} = \frac{\partial \mathcal{L}}{\partial v^\mu}(v^\mu - \gamma^\mu) - \mathcal{L}$$

It is obvious from this expression that the lagrangian energy is connection-depending. In particular, when the standard connection is used, then $\gamma^\mu = 0$, which is the fact that "hides" the explicit dependence on the connection of the energy in classical mechanics.

Notice that, since $\Theta_{\mathcal{L}}$ is π_1-semibasic, the energy function does not, in fact, depend on the extension of $\tilde{Y}$ from $Q \times \mathbf{R}$ to $\mathrm{T}Q \times \mathbf{R}$.

Another characterization of the energy can be obtained in the following way. In the geometrical description of mechanics, it is also usual to define the lagrangian energy using the vertical endomorphism. In order to achieve this result, consider the difference $\mathcal{S}^\nabla - \mathcal{V} \in \Omega^1(\mathrm{T}Q \times \mathbf{R}) \otimes \Gamma(\mathrm{T}Q \times \mathbf{R}, \mathrm{V}(\pi_1))$ and its natural contraction with $\mathrm{d}\mathcal{L}$, $\mathrm{d}\mathcal{L} \circ (\mathcal{S}^\nabla - \mathcal{V}) \in \Omega^1(\mathrm{T}Q \times \mathbf{R})$, whose local expression is

$$\mathrm{d}\mathcal{L} \circ (\mathcal{S} - \mathcal{V}) = \frac{\partial \mathcal{L}}{\partial v^\mu}(v^\mu - \gamma^\mu)\mathrm{d}t$$

Then, it is easy to verify that

$$\mathrm{E}_{\mathcal{L}}^{\nabla} = i(j^1\tilde{Y})(\mathrm{d}\mathcal{L} \circ (\mathcal{S} - \mathcal{V}) - \mathcal{L}\mathrm{d}t)$$

The 1-form

$$\mathcal{E}_{\mathcal{L}}^{\nabla} := \mathrm{d}\mathcal{L} \circ (\mathcal{S} - \mathcal{V}) - \mathcal{L}\mathrm{d}t \in \Omega^1(\mathrm{T}Q \times \mathbf{R})$$

is called the *density of lagrangian energy* associated with the lagrangian function $\mathcal{L}$ and the connection ∇ and it is a $(\pi \circ \pi_1)$-semibasic form whose local expressions is

$$\mathcal{E}_{\mathcal{L}}^{\nabla} = \left(\frac{\partial \mathcal{L}}{\partial v^\mu}(v^\mu - \gamma^\mu) - \mathcal{L} \right) \mathrm{d}t$$

At this point it is relevant to ask about the dynamical meaning of this connection-depending energies. Remember that in the standard case we have

$$X_{\mathcal{L}}(\mathrm{E}_{\mathcal{L}}) = -\frac{\partial \mathcal{L}}{\partial t} \tag{4}$$

which is a relation between the rates of variation of two functions with respect to different time actions: the action through the flow of the dynamical vector field $X_{\mathcal{L}}$ and the one given by translations in time. As we have seen above, every connection ∇ induces a lifting $j^1\tilde{Y}$ of $\dfrac{\mathrm{d}}{\mathrm{d}t}$, which is the infinitessimal generator of a new time action, and then the equivalent result to (4) is given by the following:

Theorem 1 *Let $X_{\mathcal{L}} \in \aleph(\mathrm{T}Q \times \mathbf{R})$ be the dynamical vector field (solution of the equations (3)). Then*

$$X_{\mathcal{L}}(\mathrm{E}_{\mathcal{L}}^{\nabla}) = -(j^1\tilde{Y})\mathcal{L} \tag{5}$$

(*Proof*) Notice that, since $X_{\mathcal{L}}$ is a SODE, $\mathcal{V}(X_{\mathcal{L}}) = 0$; therefore

$$
\begin{aligned}
X_{\mathcal{L}}(\mathrm{E}_{\mathcal{L}}^{\nabla}) &= i(X_{\mathcal{L}})\mathrm{d}\mathrm{E}_{\mathcal{L}}^{\nabla} = -i(X_{\mathcal{L}})\mathrm{d}\,i(j^1\tilde{Y})\Theta_{\mathcal{L}} = i(X_{\mathcal{L}})\,i(j^1\tilde{Y})\mathrm{d}\Theta_{\mathcal{L}} - i(X_{\mathcal{L}})\,\mathrm{L}(j^1\tilde{Y})\Theta_{\mathcal{L}} \\
&= i(j^1\tilde{Y})\,i(X_{\mathcal{L}})\Omega_{\mathcal{L}} + i([j^1\tilde{Y}, X_{\mathcal{L}}])\Theta_{\mathcal{L}} - \mathrm{L}(j^1\tilde{Y})\,i(X_{\mathcal{L}})\Theta_{\mathcal{L}} \\
&= -\mathrm{L}(j^1\tilde{Y})\,i(X_{\mathcal{L}})(\mathrm{d}\mathcal{L} \circ \mathcal{V} + \mathcal{L}\mathrm{d}t) = -\mathrm{L}(j^1\tilde{Y})\,i(\mathcal{V}(X_{\mathcal{L}}))\mathrm{d}\mathcal{L} - \mathrm{L}(j^1\tilde{Y})(\mathcal{L}\,i(X_{\mathcal{L}})\mathrm{d}t) \\
&= -(j^1\tilde{Y})\mathcal{L}
\end{aligned}
$$

where we have taken into account that $[j^1\tilde{Y}, X_{\mathcal{L}}]$ is π_1-vertical (because $X_{\mathcal{L}}$ is a SODE and $j^1\tilde{Y}$ is a canonical lifting [5]) and $\Theta_{\mathcal{L}}$ is a π_1-semibasic form, so $i([j^1\tilde{Y}, X_{\mathcal{L}}])\Theta_{\mathcal{L}} = 0$. ∎

Furthermore, there is a relevant relationship between the first integrals of the dynamical vector field and the lagrangian energy functions which is given in the next statement:

Proposition 4 *If $\mathcal{L}$ is a lagrangian function such that its associated Legendre transformation (see next section) is different from zero at every point, then every first integral of the dynamical vector field $X_{\mathcal{L}}$ is the energy associated to some connection.*

(*Proof*) Let ∇ be a connection such that its associated vector field Y verifies $(j^1\tilde{Y})\mathcal{L} = 0$, then it follows that $\mathrm{E}_{\mathcal{L}}^{\nabla}$ is a conserved quantity. Conversely, if $f \in \mathrm{C}^{\infty}(\mathrm{T}Q \times \mathbf{R})$ verifies $X_{\mathcal{L}}(f) = 0$, and we take the connection associated to any vector field Y such that $i(j^1\tilde{Y})\Theta_{\mathcal{L}} = -f$. But taking into account that $\Theta_{\mathcal{L}}$ is π_1-semibasic and using the definition of the Legendre transformation $F\mathcal{L}$, this equation can be written as $Y \circ F\mathcal{L} + i\left(\dfrac{\partial}{\partial t}\right)\Theta_{\mathcal{L}} = -f$, which obviously can be solved for Y if $F\mathcal{L}$ is different from zero at every point. Then we have that $\mathrm{E}_{\mathcal{L}}^{\nabla} = f$. ∎

To conclude this section, we may point out that the lagrangian formalism of non-autonomous systems does not depend on taking the standard or non-standard connections, except in order to define the energy lagrangian function, which does depend on this choice. This is the same situation as for the lagrangian formalism of field theories (see [8]).

4 Hamiltonian formalism with non-standard connections

Next we are going to consider the hamiltonian formulation of the problem, taking the bundles $\mathrm{T}^*Q \times \mathbf{R} \xrightarrow{\tau_1} Q \times \mathbf{R} \xrightarrow{\pi} \mathbf{R}$. The construction and local expressions of the canonical elements are the usual ones when the standard connection is used [9], [7], . Now we study how this situation changes when a non-standard connection is taken.

First of all, we may recall that the transition from the lagrangian formalism to the hamiltonian one is done by means of the *Legendre transformation*, which is defined for non-autonomous mechanics as in the time-independent case. So $F\mathcal{L} \colon \mathrm{T}Q \times \mathbf{R} \to \mathrm{T}^*Q \times \mathbf{R}$ is the map $F\mathcal{L} := (F\mathcal{L}_t, Id_{\mathbf{R}})$; that is, for every $(x, t) \in \mathrm{T}Q \times \mathbf{R}$,

$$F\mathcal{L}(x, t) := (F\mathcal{L}_t(x), t)$$

where $F\mathcal{L}_t\colon TQ \to T^*Q$ is the usual fiber derivative of the restriction of $\mathcal{L}$ obtained by considering its value on every fixed t, that is, $F\mathcal{L}_t(x) := F\mathcal{L}(x,t)|_{t=ctn}$. In coordinates we have

$$F\mathcal{L}^* q^\mu = q^\mu \quad , \quad F\mathcal{L}^* p_\mu = \frac{\partial \mathcal{L}}{\partial v^\mu} \quad , \quad F\mathcal{L}^* t = t$$

Then, the lagrangian function $\mathcal{L}$ is said to be *regular* iff $F\mathcal{L}$ is a local diffeomorphism and it is called *hyperregular* iff $F\mathcal{L}$ is a global diffeomorphism.

Now, let ∇ be a connection and $\tilde{\nabla} \in \Omega^1(\mathbf{R}) \otimes \aleph(Q \times \mathbf{R})$ its associated connection form. We can associate to it a canonical form defined as follows:

Definition 4 *Consider* $(\alpha, s) \in T^*Q \times \mathbf{R}$ *and* $u \in T_{(\alpha,s)}(T^*Q \times \mathbf{R})$, *then let* $\theta_\nabla \in \Omega^1(T^*Q \times \mathbf{R})$ *be defined by*

$$\theta_\nabla((\alpha, s); u) := \alpha(v_\nabla(T\tau_1(u)))$$

1. *The form θ_∇ is called the* Liouville 1-form *associated with the connection ∇.*

2. *The form $\omega_\nabla := -d\theta_\nabla$ is called the* Liouville 2-form *associated with the connection ∇.*

If t denotes the usual global coordinate in $\mathbf{R}$, (q^μ) is a local system of coordinates in Q and (q^μ, p_μ) is the canonical system of coordinates in T^*Q, the coordinate expressions of these elements are the following:

$$\tilde{\nabla} = dt \otimes \left(\frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu} \right) \equiv dt \otimes \left(\frac{\partial}{\partial t} + Y \right)$$

$$\theta_\nabla \left((\alpha, s); \frac{\partial}{\partial q^\mu} \right) = \alpha \left(v \left(\frac{\partial}{\partial q^\mu} \right) \right) = \alpha \left(\frac{\partial}{\partial q^\mu} \right) = p_\mu(\alpha)$$

$$\theta_\nabla \left((\alpha, s); \frac{\partial}{\partial p_\mu} \right) = \alpha(v(0)) = 0$$

$$\theta_\nabla \left((\alpha, s); \frac{\partial}{\partial t} \right) = \alpha \left(v \left(\frac{\partial}{\partial t} \right) \right) = \alpha(-Y) = \alpha \left(-\gamma^\mu \frac{\partial}{\partial q^\mu} \right) = -\gamma^\mu(q, s) p_\mu(\alpha)$$

(where $(q, s) = \tau_1(\alpha, s)$), therefore

$$\theta_\nabla = p_\mu dq^\mu - \gamma^\mu p_\mu dt$$

$$\omega_\nabla = dq^\mu \wedge dp_\mu + \gamma^\mu dp_\mu \wedge dt + p_\mu \frac{\partial \gamma^\mu}{\partial q^\nu} dq^\nu \wedge dt$$

Using these coordinate expressions, it is easy to prove that:

Proposition 5 $F\mathcal{L}^* \theta_\nabla = \Theta_\mathcal{L} + E_\mathcal{L}^\nabla dt.$

If ∇ is a connection in $\pi\colon Q \times \mathbf{R} \to \mathbf{R}$ and $Y \in \aleph(Q \times \mathbf{R})$ its associated vector field, let $\mathcal{Y} \in \aleph(T^*Q \times \mathbf{R})$ be any extension of $\tilde{Y}$ to $T^*Q \times \mathbf{R}$; that is, any map $\mathcal{Y}\colon T^*Q \times \mathbf{R} \to T(T^*Q \times \mathbf{R})$ making the following diagram commutative

$$
\begin{array}{ccc}
T(T^*Q \times \mathbf{R}) & \xrightarrow{T\tau_1} & T(Q \times \mathbf{R}) \\
\mathcal{Y} \uparrow \downarrow & & \downarrow \uparrow \tilde{Y} \\
T^*Q \times \mathbf{R} & \xrightarrow{\tau_1} & Q \times \mathbf{R}
\end{array}
$$

Consider now the projection $\tau_{T^*Q}\colon T^*Q \times \mathbf{R} \to T^*Q$. Let $\theta \in \Omega^1(T^*Q)$ and $\omega := -d\theta \in \Omega^2(T^*Q)$ be the canonical forms in T^*Q, and $\theta_0 = \tau_{T^*Q}^* \theta$, $\omega_0 = \tau_{T^*Q}^* \omega$. Then we can state:

Proposition 6 *The relation between the canonical forms in* $T^*Q \times \mathbf{R}$ *and the Liouville forms is given by*

$$\theta_\nabla = \theta_0 - \eta \quad , \quad \omega_\nabla = \omega_0 + \mathrm{d}\eta$$

where $\eta \in \Omega^1(T^*Q \times \mathbf{R})$ *is the form defined as*

$$\eta := \theta_0(\mathcal{Y})\mathrm{d}t \equiv H\mathrm{d}t \tag{6}$$

(*Proof*) Taking into account the coordinate expression of θ_∇, we must prove that the local expression of η is just $\eta \equiv \gamma^\mu p_\mu \mathrm{d}t$ or, what is equivalent, for every $(\alpha, s) \in T^*Q \times \mathbf{R}$ with $(q, s) = \tau_1(\alpha, s)$, that

$$H(\alpha, s) := \theta_0((\alpha, s); \mathcal{Y}) \equiv \gamma^\mu(q, s)p_\mu(\alpha)$$

Then, bearing in mind the diagram

$$
\begin{array}{ccc}
T^*Q \times \mathbf{R} & \xrightarrow{\tau_1} & Q \times \mathbf{R} \\
{\scriptstyle \tau_{T^*Q}}\downarrow & & \downarrow{\scriptstyle \pi_Q} \\
T^*Q & \xrightarrow{\tau} & Q
\end{array}
$$

we have

$$
\begin{aligned}
\theta_0((\alpha, s); \mathcal{Y}) &= \theta(\alpha; T_{(\alpha,s)}\tau_{T^*Q}\mathcal{Y}(\alpha, s)) = \alpha(T_\alpha\tau T_{(\alpha,s)}\tau_{T^*Q}\mathcal{Y}(\alpha, s)) = \alpha(T_{(\alpha,s)}(\tau \circ \tau_{T^*Q})\mathcal{Y}(\alpha, s)) \\
&= \alpha(T_{(\alpha,s)}(\pi_Q \circ \tau_1)\mathcal{Y}(\alpha, s)) = \alpha(T_{(q,s)}\pi_Q T_{(\alpha,s)}\tau_1\mathcal{Y}(\alpha, s)) = \alpha(T_{(q,s)}\pi_Q Y(q, s)) \\
&= \alpha(q; Y(q, s)) = p_\mu(\alpha)\gamma^\mu(q, s)
\end{aligned}
$$

∎

If we assume that $\mathcal{L}$ is a hyperregular lagrangian; that is, $F\mathcal{L}$ is a global diffeomorphism; then it is a simple matter of calculation to obtain the following alternative characterization of the function H

$$H = \theta_0(F\mathcal{L}_*(j^1\tilde{Y})) = F\mathcal{L}^{-1^*}((F\mathcal{L}^*\theta_0)(j^1\tilde{Y}))$$

since $F\mathcal{L}_*(j^1\tilde{Y})$ is an extension of $\tilde{Y}$ to $T^*Q \times \mathbf{R}$.

In order to introduce the dynamics we need the *hamiltonian function*. Taking into account that the physical observable corresponding to this function is the *energy*, we have:

Definition 5 *The* hamiltonian function *associated with the connection* ∇ *and the lagrangian* $\mathcal{L}$ *is the function* $h^\nabla \in C^\infty(T^*Q \times \mathbf{R})$ *such that*

$$E_\mathcal{L}^\nabla = F\mathcal{L}^* h^\nabla$$

(As one can observe, the existence of such a function is assured (at least locally) if we assume that $\mathcal{L}$ is a hyperregular (or regular) lagrangian. In other cases, under certain hypothesis on $F\mathcal{L}$, it is possible to prove that $E_\mathcal{L}^\nabla$ is also $F\mathcal{L}$-projectable).

Now we define:

Definition 6 *The* Hamilton-Cartan (1 and 2)-forms *associated with the connection* ∇ *and the hamiltonian function* h^∇ *are*

$$\Theta_\nabla := \theta_\nabla - h^\nabla \mathrm{d}t \quad ; \quad \Omega_\nabla := -\mathrm{d}\Theta_\nabla = \omega_\nabla + \mathrm{d}h^\nabla \wedge \mathrm{d}t$$

The equations of motion (*Hamilton equations*) are

$$i(X_\nabla)\Omega_\nabla = 0 \quad , \quad i(X_\nabla)\mathrm{d}t = 1$$

where $X_\nabla \in \aleph(\mathrm{T}^*Q \times \mathbf{R})$ is the dynamical vector field associated with the hamiltonian function h^∇.

At this point we must consider the influence on the dynamics of the choice of a non-standard connection in the construction of the hamiltonian formalism.

In order to do this, we take the geometrical and dynamical elements of the hamiltonian formalism when the standard connection ∇_0 in $Q \times \mathbf{R}$ is used. In this case the *standard hamiltonian function* $\mathrm{h} \in C^\infty(\mathrm{T}^*Q \times \mathbf{R})$ is defined by

$$\mathrm{E}_\mathcal{L} = \mathrm{F}\mathcal{L}^*\mathrm{h}$$

Then, the relation between the hamiltonian functions h^∇ and h is given by the following result:

Proposition 7 *Let H be the function defined by (6). Then*

$$\mathrm{h}^\nabla = \mathrm{h} - H$$

(*Proof*)   It suffices to prove it using systems of natural coordinates. Then, since $H = p_\mu \gamma^\mu$, the result holds immediately, taking into account the coordinate expressions of the functions $\mathrm{E}_\mathcal{L}^\nabla$, $\mathrm{E}_\mathcal{L}$ and the Legendre transformation. ∎

Next we construct the Liouville forms associated with ∇_0. Taking into account the proposition 6 and the fact that for the standard connection is $Y_0 = 0$, we obtain straightforwardly that these forms are just θ_0 and ω_0.

Finally, the Hamilton-Cartan forms associated with ∇_0 are defined as above, and so they are

$$\Theta_0 := \theta_0 - \mathrm{h}\mathrm{d}t \quad , \quad \Omega_0 := \omega_0 + \mathrm{d}\mathrm{h} \wedge \mathrm{d}t$$

Therefore, the equations of motion are now

$$i(X_0)\Omega_0 = 0 \quad , \quad i(X_0)\mathrm{d}t = 1$$

where $X_0 \in \aleph(\mathrm{T}^*Q \times \mathbf{R})$ is the dynamical vector field associated with the standard hamiltonian function h.

Then we have the following result:

Proposition 8 *The Hamilton-Cartan forms associated with the standard and non standard connections are the same:*

$$\Theta_0 = \Theta_\nabla \quad , \quad \Omega_0 = \Omega_\nabla$$

As a consequence the dynamical vector fields are equal: $X_\nabla = X_0$.

(*Proof*)   Taking into account the propositions 6 and 7 and the expression (6) we obtain that

$$\Theta_\nabla := \theta_\nabla - \mathrm{h}^\nabla \mathrm{d}t = \theta_0 - \eta - (\mathrm{h} - H)\mathrm{d}t = \theta_0 - \mathrm{h}\mathrm{d}t = \Theta_0$$

The remainder of the statement is a straightforward consequence of this result. ∎

In a local system of canonical coordinates we have

$$
\begin{array}{rccr}
\Theta_\nabla & = & p_\mu \mathrm{d}q^\mu - \mathrm{h}\mathrm{d}t & = \quad \Theta_0 \\
\Omega_\nabla & = & \mathrm{d}q^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\mathrm{h} \wedge \mathrm{d}t & = \quad \Omega_0 \\
X_\nabla & = & \frac{\partial}{\partial t} + \frac{\partial \mathrm{h}}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial \mathrm{h}}{\partial q^\mu}\frac{\partial}{\partial p_\mu} & = \quad X_0
\end{array}
$$

Hence we can conclude that, from the dynamical point of view in the hamiltonian formalism, no difference exists when we describe the system using the standard or non-standard connections.

As a final remark, one can easily check that $\Theta_\mathcal{L} = F\mathcal{L}^*\Theta_\nabla$. Therefore $\Omega_\mathcal{L} = F\mathcal{L}^*\Omega_\nabla$ and $X_\nabla = F\mathcal{L}_* X_\mathcal{L}$.

5 On the characterization of the energy by means of variational principles

In definition 3 we have defined the lagrangian energy function, justifying it geometrically. However, it can be considered as an "ad hoc" definition. Next we give another characterization of the energy which is based on variational principles and justifies the definition we have given.

First of all we need the following lemma:

Lemma 1 *Let $\beta \in \Omega^1(\mathrm{T}Q \times \mathbf{R})$ and $f \in \mathrm{C}^\infty(\mathrm{T}Q \times \mathbf{R})$. For every differentiable curve $\sigma\colon [a,b] \subset \mathbf{R} \to Q$, the following conditions are equivalent:*

1. $\bar{\sigma}^*(f\mathrm{d}t) = \bar{\sigma}^*\beta$.

2. $\displaystyle\int_{\bar\sigma} f\mathrm{d}t = \int_{\bar\sigma} \beta$.

(where $\bar{\sigma}\colon [a,b] \subset \mathbf{R} \to \mathrm{T}Q \times \mathbf{R}$ denotes the canonical lifting of σ).

(*Proof*) Trivially 1 ⇒ 2.

Conversely, if we suppose 1 is not true, then there exists one curve $\sigma\colon [a,b] \subset \mathbf{R} \to Q$ with $\bar{\sigma}^*(f\mathrm{d}t - \beta) \neq 0$ and hence there is $s \in [a,b]$ and a closed neighbourhood V of s in $[a,b]$ such that, taking $\gamma\colon V \to Q$ with $\gamma = \sigma|_V$, then

$$
\int_{\bar\gamma}(f\mathrm{d}t - \beta) \neq 0
$$

so 2 is false. ∎

Next we state:

Proposition 9 *The lagrangian energy function introduced in definition 3 is the unique function in $\mathrm{T}Q \times \mathbf{R}$ verifying the condition*

$$
\bar{\sigma}^*(\mathrm{E}_\mathcal{L}^\nabla \mathrm{d}t) = \bar{\sigma}^*(F\mathcal{L}^*\theta_\nabla - \mathcal{L}\mathrm{d}t)
$$

for every curve $\sigma\colon [a,b] \subset \mathbf{R} \to Q$.

(*Proof*) (Uniqueness): Let f and g be two functions verifying this condition. Obviously $\bar{\sigma}^*((f - g)\mathrm{d}t) = 0$, but $0 = \bar{\sigma}^*((f - g)\mathrm{d}t) = (f - g)(\bar{\sigma}(t))\mathrm{d}t$, for every $t \in [a, b]$. Hence, $(f - g)(\bar{\sigma}(t)) = 0$, and this implies $f - g = 0$, because every point in $\mathrm{T}Q \times \mathbf{R}$ is in the image of some curve $\bar{\sigma}$.

(Existence): From proposition 5 we obtain

$$
\begin{aligned}
\bar{\sigma}^*(\mathrm{F}\mathcal{L}^*\theta_\nabla - \mathcal{L}\mathrm{d}t) &= \bar{\sigma}^*(\Theta_\mathcal{L} + \mathrm{E}_\mathcal{L}^\nabla \mathrm{d}t + \mathcal{L}\mathrm{d}t) \\
&= \bar{\sigma}^*(\mathrm{d}\mathcal{L} \circ \mathcal{V} - \mathcal{L}\mathrm{d}t + \mathrm{E}_\mathcal{L}^\nabla \mathrm{d}t + \mathcal{L}\mathrm{d}t) = \bar{\sigma}^*(\mathrm{E}_\mathcal{L}^\nabla \mathrm{d}t)
\end{aligned}
$$

since $\bar{\sigma}^*(\mathrm{d}\mathcal{L} \circ \mathcal{V}) = 0$. So, the energy function introduced in definition 3 satisfies this condition. ∎

From the lemma 1 we obtain, for every curve $\sigma \colon [a, b] \to Q$,

$$
\int_{\bar{\sigma}} \mathrm{E}_\mathcal{L}^\nabla \mathrm{d}t = \int_{\bar{\sigma}} (\mathrm{F}\mathcal{L}^*\theta_\nabla - \mathcal{L}\mathrm{d}t)
$$

therefore

$$
\begin{aligned}
\int_{\bar{\sigma}} \mathcal{L}\mathrm{d}t &= \int_{\bar{\sigma}} (\mathrm{F}\mathcal{L}^*\theta_\nabla - \mathrm{E}_\mathcal{L}^\nabla \mathrm{d}t) = \int_{\bar{\sigma}} \mathrm{F}\mathcal{L}^*(\theta_\nabla - \mathrm{h}^\nabla \mathrm{d}t) \\
&= \int_{\bar{\sigma}} \mathrm{F}\mathcal{L}^*(\theta_0 - \mathrm{h}\mathrm{d}t) = \int_{\bar{\sigma}} \mathrm{F}\mathcal{L}^*\Theta_0 = \int_{\mathrm{F}\mathcal{L}\circ\bar{\sigma}} \Theta_0
\end{aligned}
$$

and this equality shows the equivalence between the *Hamilton principle of minimal action* (of the lagrangian formalism) and the *Hamilton-Jacobi principle* (of the hamiltonian formalism). Therefore, taking into account the proposition 9, we have to conclude that the energy (as it is defined in definition 3) is the only function that realizes the equivalence between both variational principles. This fact justifies the definition given above.

6 Conclusions

The geometric description of non-autonomous mechanics is usually given using the natural connection induced by the trivial bundle structure of the phase spaces $\mathrm{T}Q \times \mathbf{R}$ (for the lagrangian formalism) and $\mathrm{T}^*Q \times \mathbf{R}$ (for the hamiltonian formalism). We have reformulated both formalisms starting from the choice of an arbitrary connection. As a consequence of this analysis, we have shown that the geometric construction of some elements of the theory depends on this choice; namely, the lagrangian energy function of the lagrangian formalism and the hamiltonian formalism itself; in particular its geometrical structures (Liouville forms) and the hamiltonian function. This fact is hidden in the usual geometric descriptions because the natural connection associated to the trivial bundle structures is flat.

We have generalized the geometric definition of the lagrangian energy function in order to take into account the use of non-standard connections. This definition characterizes the lagrangian energy as the only function that realizes the equivalence between the Hamilton variational principle and the Hamilton-Jacobi principle.

The next step was to investigate the dynamical relevance of the choice of different connections. We have proved that the dynamics is insensitive to this choice because the Poincaré-Cartan forms of the lagrangian formalism and the Hamilton-Cartan forms of the hamiltonian one do not depend on the chosen connection.

It is worth pointing out that, from the physical point of view, the connection-dependence of the energy function can be understood as follows: for time-dependent systems, to take a non-standard connection can be interpreted as a change of the time action on $Q \times \mathbf{R}$ (which arises

from taking $H(\nabla)$ instead of $\pi^*T\mathbf{R}$ as the horizontal subbundle). Then, it is reasonable that the energy function changes in its turn, since it can be considered as the conjugate function of "time" (as is made evident in some geometrical descriptions of non-autonomous systems [7]). A significant result is the relationship between the dynamical variation of the connection-dependent energy and the variation of the lagrangian with respect to the time-action induced by the connection.

A further consequence is that, for time-dependent dynamical systems described by a given lagrangian function, we have proved that any first integral of the dynamics is the energy function for a suitable connection. Moreover, we have a means of obtaining different lagrangian energy (or hamiltonian) functions which are dynamically equivalent. This consists of taking different connections in order to construct these functions.

As a final remark, we trust that these results will help to clarify some aspects of the geometrical description of classical field theories.

Acknowledgments

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